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Reg No.:________________ Name:____________________________

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

B.Tech S1 (Special Improvement) Examination January 2021 (2019 scheme)

Course Code: MAT101

Course Name: LINEAR ALGEBRA AND CALCULUS

(2019-Scheme)

Max. Marks: 100 Duration: 3 Hours

PART A

Answer all questions, each carries 3 marks.

1 Determine the rank of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 5 & 3 & -2 \end{bmatrix}$ (3)

2 What kind of conic section is represented by the quadratic form $7{x_1}^2 + 6x_1x_2 + 7{x_2}^2 = 200$ Transform it into canonical form. (3)

3 Find the derivative of $w = x^2 + y^2$ with respect to t along the path $x = at^2$, $y = 2at$. (3)

4 Let $f(x,y) = \sqrt{3x+2y}$, find the slope of the surface $z = f(x,y)$ in the y -direction at the point (2, 5). (3)

5 Evaluate $\int_0^a\int_0^a\int_0^a (yz + xz + xy)\,dx\,dy\,dz$ (3)

6 Use polar co-ordinates to evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{\frac{3}{2}} dy\,dx$ (3)

7 Test the convergence of $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$ (3)

8 Examine whether the series convergence or not $\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$ (3)

9 Find the Maclaurin series of $\frac{1}{x+1}$ up to third degree term. (3)

10 Find the Fourier Half Range sine series of $f(x) = x$ in , $0 < x < \pi$. (3)

PART B

Answer one full question from each module, each question carries 14 marks

Module-I

11 a) Test for consistency and solve the system of equations (7)

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

b) Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ (7)

12 a) For what values of a and b do the system of equations (7)

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + az = b$$

have i) no solution ii) unique solution iii) more than one solution.

b) Find the matrix of transformation that diagonalize the matrix (7)

$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. Also write the diagonal matrix.

Module-II

13 a) If If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$. (7)

b) If the local linear approximation of a function $f(x, y, z) = xy + z^2$ at a point P is $L(x, y, z) = y + 2z - x$, find the point P. (7)

14 a) If $z = e^{xy}$, $x = 2u + v$, $y = \frac{u}{v}$ find $\frac{\partial z}{\partial u}$. (7)

b) Locate all relative extrema of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$. (7)

Module-III

15 a) Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$ by reversing the order of integration. (7)

b) Using triple integral find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x + 2y + z = 6$. (7)

16 a) Find the mass and center of gravity of the triangular lamina with vertices (0,0), (0,1) and (1,0) and density function $\delta(x,y)=xy$ (7)

b) Evaluate $\iint_R x^2\,dy\,dx$, where R is the region between $y=x$ and $y=x^2$ (7)

Module-IV

17 a) Discuss the convergence of the series (7)

$$(i)\sum_{k=1}^{\infty}\frac{k!}{k^k} \quad (ii)\sum_{k=1}^{\infty}\left(\frac{k}{k+1}\right)^{k^2}$$

b) Examine the convergence and divergence of the series (7)

$$\frac{x}{1.3}+\frac{x^2}{3.5}+\frac{x^3}{5.7}+\ldots\ldots$$

18 a) Test the convergence of $1+\frac{1.3}{3!}+\frac{1.3.5}{5!}+\frac{1.3.5.7}{7!}+\ldots\ldots\ldots$ (7)

b) Prove that the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(k-2)}{k(k+1)}$ is conditionally convergent. (7)

Module-V

19 a) Obtain Fourier series for the function $f(x) = |\sin x| \quad -\pi<x<\pi$ (7)

b) (7)

If $f(x)=\begin{cases} kx\ ;\ 0<x<\frac{\pi}{2} \\ k(\pi-x)\ ;\ \frac{\pi}{2}<x<\pi \end{cases}$ then show that

$$f(x)=\frac{4k}{\pi}\left(\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}----\right).$$

20 a) Find the Fourier cosine series of $f(x)=x^2$ in $(0,\pi)$. Hence show that (7)

$$1+\frac{1}{2^2}+\frac{1}{3^2}+---=\frac{\pi^2}{6}$$

b) Find the Fourier series for the function (7)

$$f(x) = \begin{cases} x & 0<x<1 \\ 1-x & 1<x<2 \end{cases}$$

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B-TECH S1 (Special improvement) - Jan 2021

MAT101 (Linear algebra and Calculus)

PART A

(1)

No of non zero rows = Rank of a matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 5 & 3 & -2 \end{bmatrix}$$

using Gauss elimination

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 5R_1$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 8 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

no of non zero rows = 2 = Rank (A)

(2)

$$7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$$

$$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 7-\lambda I & 3 \\ 3 & 7-\lambda I \end{bmatrix}$$

$$|A - \lambda I| = 0 \quad = (7-\lambda)^2 - 9 = 0$$

$$49 + \lambda^2 - 14\lambda - 9 = 0$$

$$= \lambda^2 - 14\lambda + 40 = 0$$

$$\lambda_2 = 4, \quad \lambda_1 = 10$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$Q = 10y_1^2 + 4y_2^2 \quad \because (Q = 200)$$

$$200 = 10y_1^2 + 4y_2^2$$

$$100 = 5y_1^2 + 2y_2^2$$

$$\frac{y_1^2}{(20)} + \frac{y_2^2}{50} = 1 \quad , \quad \text{Ellipse}$$

For expressing x in terms of y we have to find eigen vectors.

For $\lambda = 10$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 3x_2 = 0$$

$$x_1 = x_2$$

$$X_1 = \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} a$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for $\lambda = 4$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 = 0$$

$$x_1 = -x_2$$

$$X_2 = \begin{bmatrix} -a \\ a \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} a$$

$$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\frac{X_1}{|X_1|} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \qquad \frac{X_2}{|X_2|} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$x = Xy$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}} \\ \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}} \end{pmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}$$

(3) $\omega = x^2 + y^2$

$$x = at^2$$

$$\frac{dx}{dt} = 2at$$

$$\frac{dw}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dy}{dt} = 2a$$

$$\frac{dw}{dt} = 2at^2 \times 2at + 2 \times 2at + 2a$$

$$= 4a^2t^2 + 8a^2t$$

④ $f(x,y) = \sqrt{3x+2y}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{3x+2y}\right)$$

$$= \frac{1\times 2}{2\sqrt{3x+2y}} = \frac{1}{\sqrt{3x+2y}}$$

at $(2,5)$

$$= \frac{1}{\sqrt{16}} = \frac{1}{4}$$

⑤ $\int_0^a\int_0^a\int_0^a (yz+xz+xy)\,dx\,dy\,dz$

$$\int_0^a\int_0^a \left(yzx + \frac{x^2}{2}z + \frac{x^2}{2}y\right)_0^a dy\,dz = \int_0^a\int_0^a \left(ayz+\frac{a^2}{2}z+\frac{a^2}{2}y\right)dy\,dz$$

$$= \int_0^a \left(\frac{azy^2}{2} + \frac{a^2}{2}zy + \frac{a^2}{2}\frac{y^2}{2}\right)_0^a dz =$$

$$= \int_0^a \left(\frac{a^3}{2}z + \frac{a^3 z}{2} + \frac{a^4}{4}\right) dz$$

$$= \int_0^a \left[\frac{a^3}{2}\frac{z^2}{2} + \frac{a^3}{2}\frac{z^2}{2} + \frac{a^4}{4}z\right]_0^a$$

$$= \frac{a^5}{4} + \frac{a^5}{4} + \frac{a^5}{4} = \frac{3a^5}{4}$$

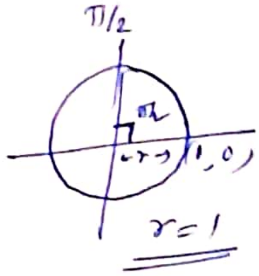
⑥ $$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$$

$x = r\cos\theta$, $y = r\sin\theta$. $x^2+y^2=r^2$, $y/x = \tan\theta$.

Region $x = -1 \longrightarrow 1$

$y = 0 \longrightarrow \sqrt{1-x^2}$

$y^2 = (1-x^2) \Rightarrow x^2+y^2 = 1$



$r = 0 \rightarrow 1$, $\theta = 0 \rightarrow \pi/2$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r^2)^{3/2}\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3\, dr\, d\theta$$

$$\int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4}\right)_0^1 d\theta = \int_{\theta=0}^{\pi/2} \frac{1}{4}\, d\theta$$

$$\frac{1}{4}\left[\theta\right]_0^{\pi/2} = \frac{1}{4}\times\frac{\pi}{2}$$

$$= \underline{\underline{\frac{\pi}{8}}}$$

7 $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$

$\sum a_k = \sum \frac{1}{(2k+3)^{17}}$

$\sum b_k = \sum \frac{1}{k^{17}}$

It is a p series with $p=17$

$p>1$ $\therefore$ Convergent

$l = \lim_{k\to\infty} \frac{a_k}{a_k} = \lim_{k\to\infty} \frac{1}{(2k+3)^{17}} \Big/ \frac{1}{2k^{17}}$

$= \lim_{k\to\infty} \frac{1/k^{17}\ \frac{1}{(2+3/k)^{17}}}{1/k^{17}}$

$= \frac{1}{2^{17}} > 0$

$\therefore$ Finite and non-zero.

8 $\sum_{k=1}^{\infty} \frac{1}{[\ln(k+1)]^k}$

Root test

$u_k = \frac{1}{[\ln(k+1)]^k}$

$(u_k)^{1/k} = \left[\frac{1}{[\ln(k+1)]^k}\right]^{1/k} = \frac{1}{\ln(k+1)}$

Now $l = \lim_{k\to\infty} (u_k)^{1/k}$.

$= \lim_{k\to\infty} \frac{1}{\ln(k+1)}$

$= \frac{1}{\infty} = 0 < 1$

$\therefore$ converges by root test

9 $f(x) = \frac{1}{1+x}$

Maclaurian Series, $x=0$

$\therefore f(0) = 1$

$f'(x) = \frac{-1}{(1+x)^2}$ $f'(0) = -1$

$f''(x) = \frac{2}{(1+x)^3}$ $f''(0) = 2.$

$f'''(x) = \frac{-6}{(1+x)^4}$ $f'''(0) = -6$

$\therefore f(x) = \frac{1}{1+x} = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$

$= 1 - x + x^2 - x^3 +$

10. $f(x) = x \quad 0 < x < \pi$.

$L = \pi$

Half range sine series.

$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\pi}$

$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$.

$= \frac{2}{\pi} \left[x \cdot \frac{\cos nx}{n} - \int 1 \times \frac{-\cos nx}{n} \right]_0^{\pi}$

$= \frac{2}{\pi} \left[x \frac{-\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$

$= \frac{2}{\pi} \left[-\frac{\pi \cos n\pi}{n} + \frac{\sin n\pi}{n^2} - \left(-\frac{x \cos 0}{n} + \frac{\sin 0}{n^2} \right) \right]$

$= \frac{2}{\pi} \left[-\frac{(-1)^n \pi}{n} + 0 - \left(-\frac{\pi}{n} + 0 \right) \right] = \frac{2}{\pi} \left[\frac{\pi}{n} - \frac{(-1)^n \pi}{n} \right]$

$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\pi/n - \frac{(-1)^n \pi}{n} \right) \sin nx$.

Module - I

11. a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ $AX = B$

$$C = [A:B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$$= \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_4$

$$= \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 1 & 4 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_1$

$$= \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_4 \to 7R_4 - 3R_2$

$$= \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$= \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank of A = rank of $[A:B]$ = n.

Unique soln.

$x + 2y - z = 3$

$-7y + 5z = -8$

$z = 4$

$x = -1$

$y = 4$

$z = 4$

b) $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0.$

$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0.$ $\lambda = 3, 1, 2$

$\underline{\lambda = 3}$

$(A - 3I) = \begin{bmatrix} -2 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix}}$$

$k = \frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{-1}$

$X = k\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$

$\underline{\lambda = 1}$

$$(A - I) = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \frac{x_1}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{-x_1}{\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix}}$$

$$k = \frac{x_1}{1} = \frac{-x_1}{-2} = \frac{x_3}{-1} \qquad X_2 = k \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\underline{\lambda = 2}$

$$[A - 2I] = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \frac{x_1}{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$k = \frac{x_1}{-1} = \frac{-x_2}{-1} = \frac{x_3}{-1} \qquad X_3 = k \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

12. a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{bmatrix}$ $\quad B = \begin{bmatrix} 6 \\ 10 \\ b \end{bmatrix}$ $\quad AX = B.$

$$C = [A : B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$$

$R_1 \rightarrow R_2 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$$

To the inequalities, if

i) when $a=3$, $b\neq 10$, $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & k \end{bmatrix}$ rank of $A \neq$ rank of $[A:B]$

System has no solution.

ii) when $a\neq 3$ and b any value. $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & k & m \end{bmatrix}$

rank of A = rank of $[A:B] = n = 3$.

System has a unique solution.

iii) when $a=3$ and $b=10$, $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank of A = rank of $[A:B] < n=3$.

System has infinite solution.

b) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ $\quad |A-\lambda I| = 0 = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$

Characteristic equation given as $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\lambda = 15, 0, 3$

For $\lambda = 15$

$[A-15I] = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$ $\quad \dfrac{x_1}{\begin{vmatrix} -8 & -4 \\ -4 & -12 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & -12 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} -6 & -8 \\ 2 & -4 \end{vmatrix}}$

$k = \dfrac{x_1}{80} = \dfrac{-x_2}{80} = \dfrac{x_3}{40}$ $\quad X_1 = k\begin{bmatrix} 80 \\ -80 \\ 20 \end{bmatrix} = k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\underline{\lambda = 0}$

$$[A - 0I] = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \qquad \frac{x_1}{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}}$$

$$k = \frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10} \qquad X_2 = k\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\underline{\lambda = 3}$

$$[A - 3I] = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \qquad \frac{x_1}{\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}}$$

$$k = \frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16} \qquad X_3 = k\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$$P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$P^{-1}AP = D = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} //$$

Module II

13. a)

$u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

Let $\frac{x}{y} = r \quad \frac{y}{z} = s \quad \frac{z}{x} = t.$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}.$$

$$= \frac{\partial u}{\partial r}\times\frac{1}{y} + \frac{\partial u}{\partial t}\times\frac{-z}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{-x}{y^2} + \frac{\partial u}{\partial s}\cdot\frac{1}{z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}.$$

$$= \frac{\partial u}{\partial s}\cdot\frac{-y}{z^2} + \frac{\partial u}{\partial t}\cdot\frac{1}{x}.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{x}{y} - \frac{\partial u}{\partial t}\cdot\frac{z}{x} - \frac{\partial u}{\partial r}\cdot\frac{x}{y} + \frac{\partial u}{\partial s}\cdot\frac{y}{z} - \frac{\partial u}{\partial s}\frac{y}{z} + \frac{\partial u}{\partial t}\frac{z}{x}.$$

$$= 0$$

b) $L(x_0, y_0, z_0) = f(x_0, y_0, z_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f_z(x_0, y_0)(z - z_0)$

$f(x_0, y_0, z_0) = xy + z^2.$

$f_x = y$

$f_y = x$

$f_z = 2z.$

$$L(x, y, z) = x_0y_0 + z_0^2 + y_0(x - x_0) + x_0(y - y_0) + 2z_0(z - z_0).$$

$$= x_0y_0 + z_0^2 + xy_0 - x_0y_0 + x_0y - x_0y_0 + 2z_0z - 2z_0^2$$

$$= -z_0^2 + 2z_0z - x_0y_0 + x_0y + xy_0 \quad \text{—①}$$

From Question $L(x, y, 1) = y + 2z - x.$

Comparing coefficients, $-z_0^2 - x_0y_0 = 0.$

$$x_0y_0 + z_0^2 = 0.$$

$2z_0 = 2. \qquad x_0 = 1 \qquad y_0 = -1.$

$z_0 = 1$

The point P is given by ordered triplet $(1,-1,1)$

14. a) $z = e^{xy}$

$x = 2u + v \qquad y = u/v$

~~$\frac{\partial z}{\partial u}$~~

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= ye^{xy} \times 2 + xe^{xy}\cdot\frac{1}{v}$$

$$= 2ye^{xy} + \frac{xe^{xy}}{v}$$

$$= 2yz + \frac{xz}{v}$$

$$= 2yz + \frac{xz(y+2)}{x}$$

$$= 2yz + (y+2)z$$

$$\frac{\partial z}{\partial u} = 3yz + 2z.$$

$\because u = vy$

$v = x - 2vy$

$x(v + 2vy) = x$

$v(y+2) = x.$

$v = \frac{x}{2+y}$

b). $f(x,y) = 3x^2 - 2xy + y^2 - 8y$

$f_x(x,y) = 6x - 2y \qquad f_y(x,y) = -2x + 2y - 8$

$f_{xx}(x,y) = 6 \qquad f_{yy}(x,y) = 2$

$f_{xy}(x,y) = -2.$

$f_x = f_y = 0 \Rightarrow 6x - 2y = 0. \qquad -2x + 2y - 8 = 0.$

$6x = 2y. \qquad -2x + 2y = 8.$

$3x = y. \qquad -x + y = 4.$

$-x + 3x = 4.$

$2x = 4$

$x = 2. \quad y = 6.$

Critical point $(2,6)$

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

$$= 6 . 2 - (-2)^2$$

$$= 12 - 4$$

$$= 8 > 0.$$

$f_{xx} = 6 > 0. \Rightarrow (2,6)$ is a relative minima.

Module - III

15. a) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx.$

$y : x \longrightarrow \infty$

$x : 0 \longrightarrow \infty$

$y : 0 \rightarrow \infty$

$x : 0 \rightarrow y.$

$$\Rightarrow \int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy.$$

$$= \int_0^\infty \left. \frac{e^{-y}}{y} x \right]_0^y dy.$$

$$= \int_0^\infty e^{-y} \, dy.$$

$$= \left. -e^{-y} \right]_0^\infty$$

$$= 1$$

b)

15.b) $V = \iiint dx\,dy\,dz$.

$z = 6 - x - 2y$.

$\therefore$ z varies from $0 \rightarrow 6 - x - 2y$.

$z = 0 \Rightarrow x + 2y = 6$.

$x = 6 - 2y$.

x varies from $0 \rightarrow 6 - 2y$.

Put $x = 0 \Rightarrow 6 = 2y$, $y = 3$.

$\therefore$ y varies from $0 \rightarrow 3$.

$$\int_0^3 \int_0^{6-2y} \int_0^{6-x-2y} dz\,dx\,dy.$$

$$= \int_0^3 \int_0^{6-2y} (6 - 2y - x)\,dx\,dy.$$

$$= \int_0^3 \left(6x - \frac{x^2}{2} - 2yx\right)_0^{6-2y} dy.$$

$$= \int_0^3 \left[6(6-2y) - \frac{(6-2y)^2}{2} - 2y(6-2y)\right] dy.$$

$$= \left[6(6y - y^2) - \frac{(6-2y)^3}{6\times -2} - 2\left[\frac{6y^2}{2} - \frac{2y^3}{3}\right]\right]_0^3$$

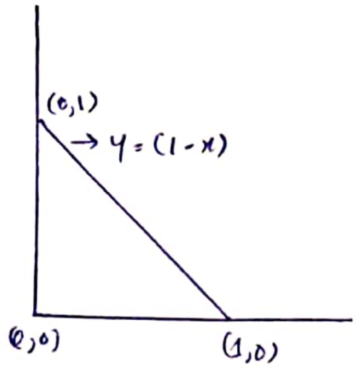
$$= \left[6(6\times 3 - 3^2) - \frac{(6-2\times 3)^3}{6\times -2} - 2\left[\frac{6\times 3^2}{2} - \frac{2\times 3^3}{3}\right]\right] - \left[6(6\times 0 - 0^2)\right] - \frac{(6-2\times 0)^3}{6\times -2} - 2\times 0$$

$= 16(18 - 9) - 2\times 9 - 18$.

$= 54 - 36$

$= \underline{18}$.

16. a)



$y : 0 \to (1-x)$

$x : 0 \to 1$

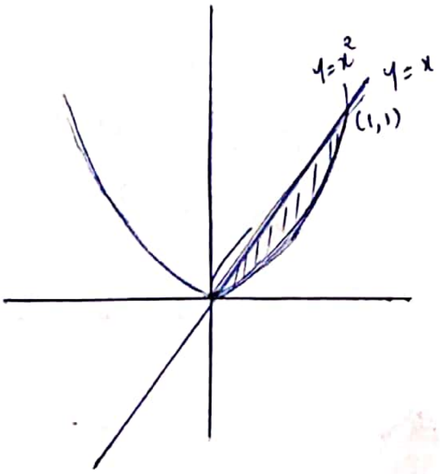
$$\int_0^1 \int_0^{1-x} xy\,dy\,dx.$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^{1-x} dx.$$

$$= \int_0^1 \frac{x(1-x)^2}{2}\,dx.$$

$$= \underline{\underline{\frac{1}{24}}}$$

b)



$y : x^2 \to x.$

$x : 0 \to 1$

$$\int_0^1 \int_{x^2}^{x} x^2\,dy\,dx$$

$$= \int_0^1 \left[x^2 y\right]_{x^2}^{x} dx$$

$$= \int_0^1 (x^3 - x^4)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= \underline{\underline{\frac{1}{20}}}$$

17) a) i) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ $\qquad u_k = \frac{k!}{k^k} \qquad u_{k+1} = \frac{(k+1)!}{(k+1)^{k+1}}$

Ratio test

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}} = \lim_{k\to\infty} \frac{\cancel{(k+1)}\, \cancel{k!}}{(k+1)^k \cancel{(k+1)}} \times \frac{k^k}{\cancel{k!}}$$

$$= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = 0 \qquad \therefore \text{ the series converges.}$$

ii) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$ applying root test $\quad u_k = \left(\frac{k}{k+1}\right)^{k^2}$

$$\lim_{k\to\infty} [u_k]^{\frac{1}{k}} = \lim_{k\to\infty} \left[\left(\frac{k}{k+1}\right)^{k^2}\right]^{1/k} = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k = 0$$

$\therefore$ the series converges.

13 a) Test the convergence of $1 + \frac{1.3}{3!} + \frac{1.3.5}{5!}$

ans)

Leaving the first term

$$a_k = \frac{1.3.5.\cdots(2k-1)}{(2k-1)!}$$

$$a_{k+1} = \frac{1.3.5\cdots(2k-1)(2k+1)}{(2k+1)!}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\cancel{1.3.5}\cdots(2k-1)(2k+1)}{(2k+1)!} \times \frac{\cancel{(2k-1)!}}{\cancel{1.3.5}\cdots\cancel{(2k-1)}}$$

$$= \lim_{k\to\infty} \frac{(2k+1)}{(2k+1)(2k)} = \lim_{k\to\infty} \frac{1}{2k} = 0$$

$\therefore$ the series ~~diverges~~ convergent by ratio test

$$= \frac{1}{\pi}\left[\frac{-(-\cos n\pi)}{1+n} - \frac{(-\cos n\pi)}{1-n} + \frac{2}{1-n^2}\right]$$

$$a_n = \left[\frac{2\cos n\pi}{1-n^2} + \frac{2}{1-n^2}\right]$$

$$= \frac{2}{\pi(1-n^2)}(\cos n\pi + 1)$$

$$= \frac{2n}{\pi(1-n^2)} \quad - (2)$$

for even $f(x)$ $n = 2n$

$$\therefore f(x) = \frac{4}{\frac{\pi}{2}} + \sum_{n=1}^{\infty} 2 \times \frac{2}{\pi(1-n^2)} \cos 2nx$$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-(2n)^2)} \cos 2nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{(4n^2-1)}$$

15) b) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k-2}{k(k+1)}$

$|u_k| = \frac{k-2}{k(k+1)} = \frac{k-2}{k^2+k}$

$|u_{k+1}| = \frac{(k+1)-2}{(k+1)((k+1)+1)} = \frac{k-1}{k^2+3k+2}$

$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \frac{k-1}{(k^2+3k+2)} \times \frac{k^2+k}{(k-2)} = \frac{k^3+\cancel{k^2}-\cancel{k^2}-k}{k^3+}$

$= \frac{1}{1} = 1 > 0$. ∴ series is not absolutely convergent by Leibnitz test

taking $a_k = \frac{k-2}{k^2+k}$ $\quad b_k = \frac{k}{k^2}$

limit comparison test

$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{k-2}{k^2+k} \times \frac{k^2}{k} = \frac{k^3\left(1-\frac{2}{k}\right)}{k^3-k^2} = \frac{1}{1} = 1$

converges or diverges equally.

$\sum_{k=1}^{\infty} b_k = \frac{k}{k^2} = \frac{1}{k}$ ∴ P series with $P = 1$ ∴ series diverges.

∴ the series converges conditionally.

19) A) obtain Fourier series for the function $f(x) = |\sin x|$

$-\pi < x < \pi$

$f(x) = |\sin x| = \sin x$

$f(-x) = |\sin(-x)| = \sin x \quad \therefore f(x)$ is even.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{x} = \frac{2}{\pi} \times 2 = \underline{\underline{\frac{4}{\pi}}} \quad - (1)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(1+n)x + \sin(1-n)x\, dx$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{\pi} \sin(1+n)x + \sin(1-n)x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{(1+n)} - \frac{\cos(1-n)}{(1-n)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} + \frac{1}{1-n} + \frac{1}{1+n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(\pi+n\pi)}{1+n} - \frac{\cos(\pi-n\pi)}{1-n} + \frac{2}{1-n^2}\right]$$

20 a)

$F(x) = x^2$

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} F(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} F(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left[\frac{\sin nx}{n}\right] - 2\pi\left[-\frac{\cos nx}{n^2}\right] + 2\left[-\frac{\sin nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$\therefore \quad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right] \quad \text{——(1)}$$

Setting $x = \pi$ in (1), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \dots\right]$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right]$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots = \frac{\pi^2}{6}$$